EQUATIONS OF GENERALIZED THERMOELASTICITY OF COSSERAT MEDIUM

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UDC 539.377

Dynamic equations of generalized thermoelasticity are derived for the Cosserat continuum, the theorem of uniqueness of solution of the problem is proved, and an expression is derived which represents the content of a theorem analogous to the reciprocal theorem.

The development of experimental physics at the present time gives rise to the need of a theoretical investigation of solid deformable media possessing more complex properties than the classical thermoelastic medium. As was noted in [1], the theory of the Cosserat continuum more accurately describes the process of deformation of granular and crystalline media. On the other hand, the law of heat conduction of Fourier forms the basis of the classical thermoelasticity, a consequence of which is an infinite rate of propagation of heat. Such a model cannot always be considered satisfactory. The account of finiteness of the propagation rate of heat can be important when investigating the effect of a short pulse of laser radiation on a thermoelastic medium.

The equations of statics of the Cosserat continuum were rigorously derived in [2] and on the principle of d'Alembert dynamic equations were obtained relative to kinematic characteristics. In [3], on the basis of equations of motion from [4], and also in [1], with the use of the principle of d'Alembert, equations of coupled thermoelasticity have been obtained for the Cosserat medium in the case of an infinite rate of heat propagation. The heat-conduction equation, with a finite propagation rate of heat taken into account, has been obtained in [5, 6] by means of introducing an additional term into the Fourier heat-conduction law. As a development of these works, there appeared a derivation of a system of equations of symmetric thermoelasticity with a finite rate of heat propagation in [7, 8]. An entire series of papers, e.g., [9-13], has been devoted to the investigation of the equations of symmetric thermoelasticity thus obtained.

A stress state arising in the Cosserat medium is described by a tensor of force stresses T and a tensor of moments stresses M, the external action on the medium is characterized by vectors of body forces X, body moments Y, surface forces F, and surface moments P. It is assumed that a temperature distribution Θ takes place. We assume that the medium is homogeneous, isotropic, and polarly symmetric. For the description of the motion of such a medium we have to introduce kinematic characteristics — a vector of displacements $\mathbf{u}$ and a vector of small rotation $\boldsymbol{\omega}$.

We consider an arbitrary volume of the Cosserat medium Φ bounded by the surface Δ. We assume that each elementary volume $d\varphi$ is characterized by the mass $\rho d\varphi$ and the inertia tensor $\mathrm{I}d\varphi$. To derive the equations of motion we have to use the equations of variation of the momentum R and the moment of the momentum K. If we consider the motion of a medium possessing a field of translational velocities V and velocities of rotation Ω, in Euler coordinates, then the elementary volume $d\varphi$ possesses the momentum

$$d\mathbf{R}_N = \rho_N \mathbf{V}_N d\varphi_N \tag{1}$$

and the moment of the momentum about the origin of the coordinates (•)O

$$d\mathbf{K}_N = \{\mathbf{I}\cdot\Omega + \zeta_{ON}\times\rho_N\mathbf{V}_N\}d\varphi_N, \tag{2}$$

where ζ_{ON} is the Euler radius vector of the point N. We assume that (•)N coincides with the center of inertia of the volume $d\varphi$.

In the analysis of the equations of variation of momentum and the moment of momentum, we have to use the rule of determination of the material derivative of the integral expres-

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 39, No. 4, pp. 716-723, October, 1980.

0022-0841/80/3904-1135$07.50

sion in Euler variables presented, e.g., in [1]. For an arbitrary volume of the medium moving with a field of translational velocities V, the following equality holds:

$$\frac{D}{Dt}\int_{\Phi} f(\zeta,\ t)d\varphi(\zeta)=\int_{\Phi}\frac{\partial f(\zeta,\ t)}{\partial t}\,d\varphi(\zeta)+\int_{\Delta} f(\zeta,\ t)\,\mathbf{V}(\zeta)\cdot\mathbf{n}(\zeta)\,d\delta(\zeta)$$
$$=\int_{\Phi}\left\{\frac{\partial f(\zeta,\ t)}{\partial t}+\mathbf{V}(\zeta)\cdot\nabla f(\zeta,\ t)+f(\zeta,\ t)\nabla\cdot\mathbf{V}(\zeta)\right\}d\varphi(\zeta). \tag{3}$$

In addition, we have to take into account the continuity equation, which is a consequence of conservation of the mass:

$$\frac{\partial\rho}{\partial t}+\mathbf{V}\cdot\nabla\rho+\rho\nabla\cdot\mathbf{V}=0. \tag{4}$$

The equations of variation of momentum and the moment of momentum have the form

$$\frac{D}{Dt}\int_{\Phi}\rho(\zeta)\mathbf{V}(\zeta)\,d\varphi(\zeta)=\int_{\Phi}\mathbf{X}(\zeta)d\varphi(\zeta)+\int_{\Delta}\mathbf{n}(\zeta)\cdot\mathbf{T}(\zeta)\,d\delta(\zeta);$$
$$\frac{D}{Dt}\int_{\Phi}\{\mathbf{I}(\zeta)\cdot\Omega(\zeta)+\zeta\times\rho(\zeta)\,\mathbf{V}(\zeta)\}\,d\varphi(\zeta)=\int_{\Phi}\{\zeta\times\mathbf{X}(\zeta)+\mathbf{Y}(\zeta)\}d\varphi(\zeta)+\int_{\Delta}\{\mathbf{n}(\zeta)\cdot\mathbf{M}(\zeta)+\zeta\times[\mathbf{n}(\zeta)\cdot\mathbf{T}(\zeta)]\}\,d\delta(\zeta). \tag{5}$$

Having used the differentiation rule (3), the continuity equation (4) and an analogous equation relative to I, and also the theorem of Ostrogradskii—Gauss, in view of arbitrariness of the volume Φ we obtain the local equations of motion

$$\nabla\cdot\mathbf{T}(\zeta,\ t)+\mathbf{X}(\zeta,\ t)=\rho\frac{D\mathbf{V}(\zeta,\ t)}{Dt};$$
$$\nabla\cdot M(\zeta,\ t)+\mathbf{Y}(\zeta,\ t)-2\mathbf{a}^T(\zeta,\ t)=\mathbf{I}\cdot\frac{D\Omega(\zeta,\ t)}{Dt}. \tag{6}$$

Here $\mathbf{a}^T$ is a vector accompanying the tensor T [14].

It should be noted that if we neglect the terms of the second order of smallness, as is always done in the linear theory of elasticity, then the equations assume the form

$$\nabla\cdot\mathbf{T}(\mathbf{r},\ t)+\mathbf{X}(\mathbf{r},\ t)=\rho\frac{\partial^2\mathbf{u}(\mathbf{r},\ t)}{\partial t^2};$$
$$\nabla\cdot\mathbf{M}(\mathbf{r},\ t)+\mathbf{Y}(\mathbf{r},\ t)-2\mathbf{a}^T(\mathbf{r},\ t)=\mathbf{I}\cdot\frac{\partial^2\omega(\mathbf{r},\ t)}{\partial t^2}, \tag{7}$$

where **r** is the Lagrange vector of a point.

For the derivation of the complete system of equations of thermoelasticity, we require the defining equations. These equations are presented in [1] and have the form

$$\mathbf{T}=2\mu\gamma^{+}+2\alpha\gamma^{-}+(\lambda\gamma^{+}\cdot\cdot\mathbf{E}-\nu\Theta_0\vartheta)\mathbf{E};$$
$$\mathbf{M}=2\gamma\varkappa^{+}+2\varepsilon\varkappa^{-}+\beta(\varkappa^{+}\cdot\cdot\mathbf{E})\,\mathbf{E}; \tag{8}$$
$$s=\nu\gamma^{+}\cdot\cdot\mathbf{E}+m\Theta_0\vartheta,$$

where

$$\varkappa=\nabla\omega;\quad \gamma=\nabla\mathbf{u}+\eta_{\omega}^{-}. \tag{9}$$

Here the following notation has been introduced: η_{ω}^{-} is an antisymmetric tensor for which ω is the accompanying vector; μ, α, λ, ν, γ, ε, β, m are material constants, introduced in [1], which characterize the mechanical and thermophysical properties of the medium. By the index plus we have marked the symmetric part of the corresponding tensor, and by minus the antisymmetric part. For the derivation of the heat-conduction equation we require the third of the defining equations (8), the equation of the entropy balance [15]

$$\Theta\dot{s}=-\nabla\cdot\mathbf{q}+w \tag{10}$$

and the generalized Fourier law [5]

$$\tau_0\dot{\mathbf{q}}+\mathbf{q}=-\Theta_0k\nabla\vartheta. \tag{11}$$

Eliminating from these three equations s and **q**, we obtain the generalized heat-conduction equation

$$k\nabla^2\vartheta - \tau_0\Theta_0 m\ddot{\vartheta} - \Theta_0 m\dot{\vartheta} - \tau_0\nu\nabla\cdot\ddot{\mathbf{u}} - \nu\nabla\cdot\dot{\mathbf{u}} = -\frac{w}{\Theta_0} - \frac{\tau_0\dot{w}}{\Theta_0}. \tag{12}$$

Having used Eqs. (6) and the first two from (8), we obtain the system of equations relative to the kinematic characteristics **u** and ω:

$$\begin{aligned}&(\mu+\alpha)\nabla^2\mathbf{u} + (\mu-\alpha+\lambda)\nabla\nabla\cdot\mathbf{u} + 2\alpha\nabla\times\boldsymbol{\omega} + \mathbf{X} - \nu\Theta_0\nabla\vartheta = \rho\ddot{\mathbf{u}};\\ &(\gamma+\varepsilon)\nabla^2\boldsymbol{\omega} + (\gamma-\varepsilon+\beta)\nabla\nabla\cdot\boldsymbol{\omega} + 2\alpha\nabla\times\mathbf{u} - 4\alpha\boldsymbol{\omega} + \mathbf{Y} = \mathbf{I}\cdot\ddot{\boldsymbol{\omega}}.\end{aligned} \tag{13}$$

Equations (13) and (12) represent the complete system of equations of generalized thermoelasticity for the Cosserat medium.

We consider a body occupying a volume Φ^* bounded by the surface Δ^* for given vectors of body forces $\mathbf{X}^*$, body moments $\mathbf{Y}^*$, surface forces $\mathbf{F}^*$, and surface moments $\mathbf{P}^*$. We assume that inside the volume there takes place heat release with density w^*, while on the surface a temperature H^* is given.

Having multiplied the first of Eqs. (7) scalarly by **u**, and the second by ω, having integrated over the volume, and having used the Ostrogradskii–Gauss theorem, the first two defining equations (8), and the boundary conditions, we obtain the first energy equation

$$A + L - \dot{W} - \dot{\Pi} + \nu\Theta_0\int_{\Phi^*}\vartheta\nabla\cdot\dot{\mathbf{u}}\,d\varphi = 0, \tag{14}$$

where

$$\begin{gathered}\int_{\Delta^*}(\dot{\mathbf{u}}\cdot\mathbf{F}^* + \dot{\boldsymbol{\omega}}\cdot\mathbf{P}^*)\,d\delta = A; \quad \int_{\Phi^*}(\dot{\mathbf{u}}\cdot\mathbf{X}^* + \dot{\boldsymbol{\omega}}\cdot\mathbf{Y}^*)\,d\varphi = L;\\ \int_{\Phi^*}\{\rho\dot{\mathbf{u}}^2 + (\mathbf{I}\cdot\dot{\boldsymbol{\omega}})\cdot\dot{\boldsymbol{\omega}}\}\,d\varphi = 2W;\\ \int_{\Phi^*}\Big\{\gamma(\boldsymbol{\varkappa}^+\cdot\cdot\boldsymbol{\varkappa}^+) + \varepsilon(\boldsymbol{\varkappa}^-\cdot\cdot\boldsymbol{\varkappa}^-) + \mu(\boldsymbol{\gamma}^+\cdot\cdot\boldsymbol{\gamma}^+) + \alpha(\boldsymbol{\gamma}^-\cdot\cdot\boldsymbol{\gamma}^-) + \frac{\beta}{2}(\nabla\cdot\boldsymbol{\omega})^2 + \frac{\lambda}{2}(\nabla\cdot\mathbf{u})^2\Big\}\,d\varphi = \Pi.\end{gathered} \tag{15}$$

Having multiplied (12) by ϑ and integrated over the volume, we obtain, with the assumption about the distribution of the temperature H^* being given on the surface, the second energy equation

$$Q + U + \mathrm{X} = \dot{\Gamma} + \nu\Theta_0\int_{\Phi^*}\vartheta\nabla\cdot\dot{\mathbf{u}}\,d\varphi. \tag{16}$$

The following notation has been used here:

$$\begin{gathered}\int_{\Delta^*}k\Theta_0 H^*\mathbf{n}\cdot\nabla\vartheta\,d\delta = Q; \quad \int_{\Phi^*}(w^* + \tau_0\dot{w}^*)\vartheta\,d\varphi = U;\\ \frac{m\Theta_0^2}{2}\int_{\Phi^*}\vartheta^2\,d\varphi = \Gamma;\\ \Theta_0 k\int_{\Phi^*}(\nabla\vartheta)^2\,d\varphi + \tau_0 m\Theta_0^2\int_{\Phi^*}\ddot{\vartheta}\vartheta\,d\varphi + \nu\tau_0\Theta_0\int_{\Phi^*}\vartheta\nabla\cdot\ddot{\mathbf{u}}\,d\varphi = -\mathrm{X}.\end{gathered} \tag{17}$$

Eliminating from (14) and (16) the common term, we obtain the fundamental energy equation

$$A + L + Q + U + \mathrm{X} = \dot{W} + \dot{\Pi} + \dot{\Gamma}. \tag{18}$$

Equation (18) can be used for the proof of uniqueness of the solution of the problem. We assume that $\gamma, \varepsilon, \alpha, \mu, \beta, \lambda, m \geq 0$. If we assume that the problem has two solutions: $\mathbf{u}_1, \omega_1, \vartheta_1$ and $\mathbf{u}_2, \omega_2, \vartheta_2$, then $\omega_0 = \omega_1 - \omega_2$, $\mathbf{u}_0 = \mathbf{u}_1 - \mathbf{u}_2$, $\vartheta_0 = \vartheta_1 - \vartheta_2$ is also a solution and at the same time satisfies the homogeneous boundary conditions and the zero initial conditions. The fundamental energy equation in this case has the form

$$\dot{W}_0 + \dot{\Pi}_0 + \dot{\Gamma}_0 = \mathrm{X}_0. \tag{19}$$

Integrating (19) over time in the interval from 0 to t, we obtain, taking into account the initial conditions,

$$W_0(t)+\Pi_0(t)+\Gamma_0(t)=\int\limits_0^t X_0 dt. \tag{20}$$

It should be mentioned that the terms located on the left side of the equation are nonnegative according to definitions (15) and (17). We consider the integrand function on the right side of (20). Transforming it in accordance with the equation of entropy balance (10) and the defining equation for entropy (8), we obtain

$$-X_0=\Theta_0 k\int\limits_{\Phi^*}(\nabla\vartheta_0)^2\,d\varphi+\tau_0 m\Theta_0^2\int\limits_{\Phi^*}\ddot{\vartheta}_0\vartheta_0 d\varphi+\nu\tau_0\Theta_0\int\limits_{\Phi^*}\vartheta_0\nabla\cdot\ddot{\mathbf{u}}\,d\varphi$$

$$=\Theta_0 k\int\limits_{\Phi^*}(\nabla\vartheta_0)^2\,d\varphi+\tau_0\Theta_0\int\limits_{\Phi^*}\ddot{s}_0\vartheta_0\,d\varphi=\Theta_0 k\int\limits_{\Phi^*}(\nabla\vartheta_0)^2\,d\varphi-\tau_0\int\limits_{\Phi^*}\vartheta_0\nabla\cdot\dot{\mathbf{q}}_0\,d\varphi-\tau_0\Theta_0\int\limits_{\Phi^*}\dot{s}_0\dot{\vartheta}_0\,d\varphi-\tau_0\Theta_0\int\limits_{\Phi^*}\ddot{s}_0\vartheta_0^2 d\varphi. \tag{21}$$

Since we consider the linear theory of thermoelasticity, in the investigation of expression (21) we have to retain only the first-order terms. Consequently, the expression for X_0 can be transformed in the following manner in accordance with the generalized Fourier law (11):

$$-X_0=\Theta_0 k\int\limits_{\Phi^*}(\nabla\vartheta_0)^2\,d\varphi-\tau_0\int\limits_{\Phi^*}\vartheta_0\nabla\cdot\dot{\mathbf{q}}_0\,d\varphi=\Theta_0 k\int\limits_{\Phi^*}(\nabla\vartheta_0)^2\,d\varphi-\tau_0$$

$$\times\int\limits_{\Phi^*}\nabla\cdot(\vartheta_0\dot{\mathbf{q}}_0)\,d\varphi+\tau_0\int\limits_{\Phi^*}\dot{\mathbf{q}}_0\cdot\nabla\vartheta_0\,d\varphi=\Theta_0 k\int\limits_{\Phi^*}(\nabla\vartheta_0)^2 d\varphi-\tau_0$$

$$\times\int\limits_{\Delta^*}\vartheta_0\mathbf{n}\cdot\dot{\mathbf{q}}_0\,d\delta+\tau_0\int\limits_{\Phi^*}\dot{\mathbf{q}}_0\cdot\nabla\vartheta_0\,d\varphi=\Theta_0 k\int\limits_{\Phi^*}(\nabla\vartheta_0)^2 d\varphi\quad-\int\limits_{\Phi^*}\mathbf{q}_0\cdot\nabla\vartheta_0\,d\varphi-\Theta_0 k\int\limits_{\Phi^*}(\nabla\vartheta_0)^2 d\varphi=\int\limits_{\Phi^*}\mathbf{q}_0\cdot\nabla\vartheta_0\,d\varphi. \tag{22}$$

As is known from the thermodynamics of irreversible processes [15],

$$-(\mathbf{q}\cdot\nabla\vartheta)\,\Theta^{-2}\Theta_0\geqslant 0. \tag{23}$$

Consequently, Eq. (20) is possible only in the case where

$$\mathbf{u}_0=0;\quad \boldsymbol{\omega}_0=0;\quad \vartheta_0=0. \tag{24}$$

Thus,

$$\mathbf{u}_1=\mathbf{u}_2;\quad \boldsymbol{\omega}_1=\boldsymbol{\omega}_2;\quad \vartheta_1=\vartheta_2, \tag{25}$$

and the solution of the boundary-value problem of thermoelasticity for the Cosserat medium in the case of a finite rate of heat propagation is unique. It should be noted that for $I=0$, $Y=0$, $P=0$, $\gamma=0$, $\varepsilon=0$ the proof just presented is transformed into the proof of uniqueness of the solution of the boundary-value problem of generalized thermoelasticity for a symmetric medium [8]. At the same time, it seems that the proof of uniqueness of the solution of a such problem presented in [8] contains an inaccuracy, since it rests on the assumption $\ddot{s}_0\vartheta_0\geq 0$ which, in our view, is not substantiated.

As is seen from the analysis of the expressions for X_0, (21) and (22), it can be shown that the right side of the expression (1.73) in [8] is negative in the aggregate, which is sufficient from the proof of the theorem.

We consider two systems of acting forces, moments and heat sources, and also their corresponding kinematic and force variables and temperatures (primed and nonprimed). Going over in Eqs. (7), (8), and (12) to the Laplace transforms, we can obtain the following two equations:

$$\mathbf{T}_p\cdot\cdot\boldsymbol{\gamma}_p'-\mathbf{T}_p'\cdot\cdot\boldsymbol{\gamma}_p+\mathbf{M}_p\cdot\cdot\boldsymbol{\varkappa}_p'-\mathbf{M}_p'\cdot\cdot\boldsymbol{\varkappa}_p=\nu\Theta_0(\vartheta_p'\mathbf{E}\cdot\cdot\boldsymbol{\gamma}_p-\vartheta_p\mathbf{E}\cdot\cdot\boldsymbol{\gamma}_p');$$

$$k(\vartheta_p'\nabla^2\vartheta_p-\vartheta_p\nabla^2\vartheta_p)-\nu p(1+p\tau_0)(\vartheta_p'\nabla\cdot\mathbf{u}_p-\vartheta_p\nabla\cdot\mathbf{u}_p')=(1+p\tau_0)\Theta_0^{-1}(w_p'\vartheta_p-w_p\vartheta_p'). \tag{26}$$

Integrating these equations over the volume and transforming them by means of the Ostrogradskii–Gauss theorem, the equations of motion, andthe boundary conditions, after elimination of the common term we obtain the equation

$$p(1+p\tau_0)\left\{\int\limits_{\Phi^*}[\mathbf{u}_p'\cdot\mathbf{X}_p-\mathbf{u}_p\cdot\mathbf{X}_p'+\boldsymbol{\omega}_p'\cdot\mathbf{Y}_p-\boldsymbol{\omega}_p\cdot\mathbf{Y}_p']\,d\varphi+\int\limits_{\Delta^*}[\mathbf{P}_p\cdot\boldsymbol{\omega}_p'-\mathbf{P}_p'\cdot\boldsymbol{\omega}_p+\mathbf{F}_p\cdot\mathbf{u}_p'-\mathbf{F}_p'\cdot\mathbf{u}_p]\,d\delta\right\}$$

$$+(1+p\tau_0)\int\limits_{\Phi^*}(w_p'\vartheta_p-w_p\vartheta_p')\,d\varphi=\Theta_0 k\int\limits_{\Delta^*}\mathbf{n}\cdot[H_p'\nabla\vartheta_p-H_p\nabla\vartheta_p']\,d\delta. \tag{27}$$

Expression (27) constitutes the content of the reciprocal theorem relative to the variables transformed according to Laplace. Having used the theorem on convolution, it is not difficult to obtain the final form of the theorem

$$\int_0^t \Big\{ \int_{\Phi^*} \Big[\frac{\partial \mathbf{u}'(\mathbf{r}, \tau)}{\partial \tau} \cdot \mathbf{X}(\mathbf{r}, t-\tau) - \frac{\partial \mathbf{u}(\mathbf{r}, \tau)}{\partial \tau} \cdot \mathbf{X}'(\mathbf{r}, t-\tau) + \frac{\partial \boldsymbol{\omega}'(\mathbf{r}, \tau)}{\partial \tau} \cdot \mathbf{Y}(\mathbf{r}, t-\tau) - \frac{\partial \boldsymbol{\omega}(\mathbf{r}, \tau)}{\partial \tau} \cdot \mathbf{Y}'(\mathbf{r}, t-\tau) \Big]$$

$$+ \tau_0 \Big[\frac{\partial^2 \mathbf{u}'(\mathbf{r}, \tau)}{\partial \tau^2} \cdot \mathbf{X}(\mathbf{r}, t-\tau) - \frac{\partial^2 \mathbf{u}(\mathbf{r}, \tau)}{\partial \tau^2} \cdot \mathbf{X}'(\mathbf{r}, t-\tau) + \frac{\partial^2 \boldsymbol{\omega}(\mathbf{r}, \tau)}{\partial \tau^2} \cdot \mathbf{Y}(\mathbf{r}, t-\tau) - \frac{\partial^2 \boldsymbol{\omega}(\mathbf{r}, \tau)}{\partial \tau^2} \cdot \mathbf{Y}'(\mathbf{r}, t-\tau) \Big]$$

$$- \vartheta'(\mathbf{r}, \tau) w(\tau, t-\tau) + \vartheta(\mathbf{r}, \tau) w'(\mathbf{r}, t-\tau) - \tau_0 \Big[\frac{\partial \vartheta'(\mathbf{r}, \tau)}{\partial \tau} w(\mathbf{r}, t-\tau)$$

$$- \frac{\partial \vartheta(\mathbf{r}, \tau)}{\partial \tau} w'(\mathbf{r}, t-\tau) \Big] d\varphi + \int_{\Delta^*} \Big[\frac{\partial \mathbf{u}'(\mathbf{r}, \tau)}{\partial \tau} \cdot \mathbf{F}(\mathbf{r}, t-\tau)$$

$$- \frac{\partial \mathbf{u}(\mathbf{r}, \tau)}{\partial \tau} \cdot \mathbf{F}'(\mathbf{r}, t-\tau) + \frac{\partial \boldsymbol{\omega}'(\mathbf{r}, \tau)}{\partial \tau} \cdot \mathbf{P}(\mathbf{r}, t-\tau) - \frac{\partial \boldsymbol{\omega}(\mathbf{r}, \tau)}{\partial \tau} \cdot \mathbf{P}'(\mathbf{r}, t-\tau) + \tau_0 \Big[\frac{\partial^2 \mathbf{u}'(\mathbf{r}, \tau)}{\partial \tau^2} \cdot \mathbf{F}(\mathbf{r}, t-\tau)$$

$$- \frac{\partial^2 \mathbf{u}(\mathbf{r}, \tau)}{\partial \tau^2} \cdot \mathbf{F}'(\mathbf{r}, t-\tau) + \frac{\partial^2 \boldsymbol{\omega}'(\mathbf{r}, \tau)}{\partial \tau^2} \cdot \mathbf{P}(\mathbf{r}, t-\tau)$$

$$- \frac{\partial^2 \boldsymbol{\omega}(\mathbf{r}, \tau)}{\partial \tau^2} \cdot \mathbf{P}'(\mathbf{r}, t-\tau) \Big] - \Theta_0 k [H'(\mathbf{r}, t-\tau) \nabla \vartheta(\mathbf{r}, \tau)$$

$$- H(\mathbf{r}, t-\tau) \nabla \vartheta'(\mathbf{r}, \tau)] \cdot \mathbf{n}(\mathbf{r}) \Big] d\delta(\mathbf{r}) \Big\} d\tau = 0. \qquad (28)$$

It is not difficult to see that for $\tau_0 = 0$ the expression just found is transformed into the result obtained in [1, 3] for the Cosserat continuum in the case of an infinite rate of propagation of heat, while for $\mathbf{P} = 0$, $\mathbf{Y} = 0$ it is transformed into the expression for the reciprocal theorem for symmetric thermoelastic medium with a finite rate of heat propagation [8]. Here we have to bear in mind that in the derivation of expressions (26)-(28) we assumed that the initial conditions were zero conditions.

NOTATION

Θ, absolute temperature; $d\delta$, surface element; ζ, Euler radius vector of a point; $\mathbf{r}$, Lagrange radius vector of a point; $\mathbf{n}$, vector of the external normal to the surface; E, unit tensor; s, entropy per unit volume; $\mathbf{q}$, vector of heat flux; w, density of volumetric heat release; $\vartheta = (\Theta - \Theta_0)\Theta_0^{1}$, relative deviation of temperature from the initial value; Θ_0, initial absolute temperature of the medium; γ, asymmetric strain tensor; $\varkappa$, tensor of flexure and torsion; τ_0, constant characterizing the rate of heat propagation; k, coefficient of thermal conductivity; A, mechanical power of external surface forces; L, mechanical power of external body forces; W, kinetic energy of strain; Π, potential energy of strain; $\mathbf{X}$, dissipation function; Γ, temperature potential; U, thermal analog of the power of internal sources; Q, thermal analog of the power of the surface of sources.

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KOCHIN—LOITSYANSKII METHOD IN FREE CONVECTION PROBLEMS

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Freely convective heat transfer is computed by the Kochin—Loitsyanskii method on a vertical plate whose temperature is variable.

Integral methods used for approximate computations of freely convective heat transfer are based on the approximation of the exact velocity and temperature profiles in the boundary layer on polynomials or other functions (exponentials, for instance). The boundary-layer "thickness" and longitudinal velocity scale introduced provisionally are determined from the solution of the integral equations. A method using a particular class of exact solutions, a one-parameter family of self-similar profiles [1], exists in boundary-layer theory. A strictly defined quantity, the thickness of the momentum loss, which is a functional of the solution of the boundary-layer equations, is used as the scale of the transverse coordinate. We use this idea to compute freely convective heat transfer.

Let us consider free convection on a vertical plate with a given wall temperature ϑ_w. We assume that the energy dissipation and work of compression are negligibly small. Integral equations in a freely convective boundary layer have the form [2]

$$\frac{d}{dx}\int_0^\infty u^2 dy = g\beta\int_0^\infty \vartheta dy - \nu \left.\frac{\partial u}{\partial y}\right|_{y=0},$$
$$\frac{d}{dx}\int_0^\infty u\vartheta dy = -\frac{\nu}{\mathrm{Pr}}\left.\frac{\partial \vartheta}{\partial y}\right|_{y=0}. \quad (1)$$

Let us introduce the transformation scale

$$h(x) = \left(\int_0^\infty u\vartheta dy\right)^2 \Big/ \left(\vartheta_w^2 \int_0^\infty u^2 dy\right),$$
$$U(x) = \vartheta_w \int_0^\infty u^2 dy \Big/ \int_0^\infty u\vartheta dy, \quad z = h^2/\nu \quad (2)$$

and substituting dimensionless functions in the equations, we obtain

M. I. Kalinin Leningrad Polytechnic Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 39, No. 4, pp. 724-727, October, 1980. Original article submitted September 24, 1979.

 0022-0841/80/3904-1140$07.50